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ON THE BIFURCATION OF SEPARATRICES IN THE PROBLEM OF STABILITY LOSS OF AUTO-OSCILLATIONS NEAR 1:4 RESONANCE*

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The problem of bifurcation of periodic motion near a $1:n$ resonance with $n=4$ is considered. The problem is reduced to the analysis of some complex differential equation dependent on a complex parameter ε. When ε is moved around zero, a sequence of bifurcations, generally of codimension 1, arise in the equation. Under certain conditions imposed on the coefficients of the equation (conditions of degeneracy) codimension 2 and higher may appear in the bifurcation sequence. In the present paper the degeneration conditions associated with the rearrangement of separatrices are determined, which made it possible to define all bifurcation sequences of the general form.

A similar problem was previously solved in /1,2/ for $n\neq4$; because all bifurcations of codimension 2 proved to be local, it was possible to define them by explicit algebraic equations. The 1:4 resonance is distinguished by that a part of bifurcations of codimension 2 is associated with the rearrangement of separatrices. In investigations of such bifurcations both the analytic and numerical methods are used.

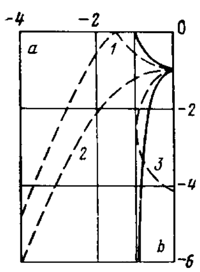

Fig.1

We consider the complex differential equation

$$(0.1)\; z^{\cdot} = e^{i\alpha} z + Az\,|z|^2 + \bar{z}^3,\quad 0\leqslant\alpha<2\pi,\; A=a+ib \tag{0.1}$$

A parametric picture of Eq. (0.1) is shown in Fig.1 in the form of partitioning of the plane of parameter A by curves that correspond to bifurcations of codimension 2 (owing to the picture symmetry, only the region of $\operatorname{Re} A\leqslant0$ and $\operatorname{Im} A\leqslant0$) is shown). Curves which correspond to nonlocal bifurcations, i.e. to the formation of a multiple separatrix cycle and of cycles consisting of saddle-node separatrices were obtained numerically. They are shown in Fig.1 by dash lines (the shape of these curves was predicted in /2/). Sequences of bifurcations in all regions of Fig.1 were defined earlier (**).

1. Bifurcations.

In describing bifurcations we assume, wherever this is essential, that $\operatorname{Re} A\leqslant0, \operatorname{Im} A\leqslant0$, and angle α increases; the events of "birth" and "disappearance" will not be distinguished. The variation of α in Eq. (0.1) results in the following one-parameter bifurcations.

1) Change of the zero singular point stability accompanied by the birth of a "central" limit cycle.

2) Appearance of four "peripheral" multiple singular points of the saddle-node type. The parameters are related by the formula

$$|\,b\cos\alpha - a\sin\alpha\,| = 1 \tag{1.1}$$

Note that saddle-nodes may appear outside the central limit cycle, on it, or inside it (Fig.2). After birth the saddle-nodes break up into four saddles and four nodes.

3) Closing of peripheral saddle separatrices, i.e. formation of a separatrix cycle (Fig. 3,a,b). Such cycle, depending on its stability, may absorb the existing central cycle or generate another central cycle.

4) Merging of two central cycles.

5) Stability change of four peripheral foci accompanied by generation of four peripheral cycles.

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**) F. S. Berezovskaia and A. I. Khibnik, On the problem of auto-oscillation bifurcations near 1:4 resonance (investigation of the model equation). Preprint. Pushchino, 1979.

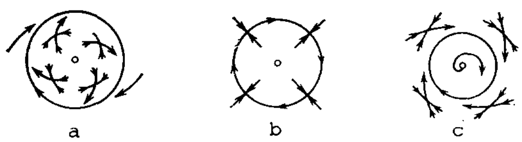

Fig.2

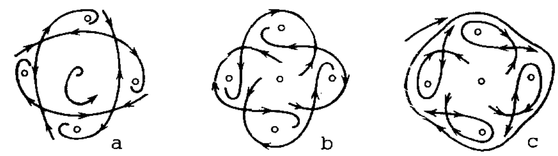

Fig.3

6) Formation of separatrix loops of each of the four saddles (Fig.3,c) accompanied by absorption of perimeter limit cycles.

Note that bifurcations 2, 3, 5, and 6 are of codimension 1 owing to the invariance of Eq.(0.1) to phase space turns by the angle $\pi/2$.

Different sequences of these bifurcations are realized with different A. Regions of the plane of parameter A (Fig.1) correspond to bifurcations sequences of the same type. The boundary curves separate one chain of events (bifurcations) from another. For instance, multiple singular points do not at all appear in region $|A|<1$, while being necessarily present in all remaining regions. Region boundaries can be more exactly described as projections on the plane of parameter A of parametric curves that correspond to bifurcations of codimension 2.

We recall the exact formulas for known boundary curves /1,2/. For α satisfying formula (1.1) multiple peripheral singular points on line $a^2+b^2=1$ are generated at the infinitely distant part of the z-plane. At $\alpha=\pm\pi/2$ two bifurcations of codimension 1 occur simultaneously on line $|a|=1$, viz. peripheral singular points and the change of stability of the central singular point. When relations (1.1) are satisfied, multiple peripheral singular points have two zero eigenvalues on curves $b=\pm(1+a^2)/\sqrt{1-a^2}$. When $\alpha=\pm\pi/2$, Eq.(0.1) is Hamiltonian on line $a=0$.

The remaining boundary curves are related to the rearrangement of separatrices. We pass to a detailed description of these.

2. Parametric curves associated with separatrix rearrangements.

With nonlocal bifurcations of codimension 2, Eq.(2) can possibly contain: closure of saddle-node separatrices in a separatrix cycle and a multiplicity of the separatrix cycle formed by separatrices of saddles.

To the first bifurcation correspond two parametric curves: one of which is the boundary between regions in which saddle-nodes appear outside the central cycle and on it (curve *1*), and the other on and inside the central cycle (curve *2*).

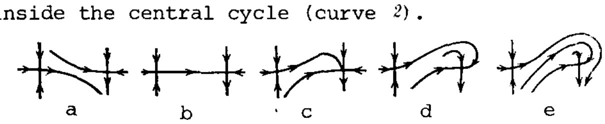

Fig.4

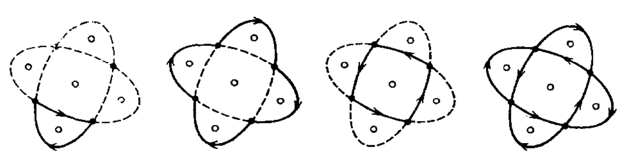

Fig.5

The line of multiple separatrix cycles (curve *3*) separates the regions in which the separatrix cycle stability is different.

Below, curves *1–3* are called nonlocal bifurcation lines.

Curves *1* and *2*. Generally the multiple singular points (saddle-nodes) can appear outside, on and inside the central limit cycle (Fig.2).

Among the separatrices of each newly generated saddle-nodes two usually correspond to a nonzero eigenvalue (and separate the node and saddle sectors) and one to the zero eigenvalue (and separates two saddle sectors). Let us consider the relative position of separatrices of two adjacent saddle-nodes. In Fig.4,a,c,e are roughly shown the cases of separatrix disposition, while in Fig.4,b,d appear bifurcation patterns consisting of the coincidence of the singular separatrix of one saddle-node and one of the conventional separatrices of the other.

These bifurcations are of codimension 2 and determine the sought boundaries between regions.

Curve *3*. Formation of a multiple separatrix cycle is possible at the closure of peripheral saddle separatrics. The necessary condition of a separatrix cycle multiplicity is its neutrality.

A cycle formed by separatrices of saddles $x_1, \ldots, x_m$ is called neutral if the eigenvalues $\lambda_i > 0$, $-\mu_i < 0$ of saddles x_i satisfy the relation

$$\delta \equiv \prod_{i=1}^{m} \frac{\lambda_i}{\mu_i} = 1 \tag{2.1}$$

(when $\delta < 1$ the separatrix cycle is stable, when $\delta > 1$ it is unstable /3/).

For Eq.(0.1) condition (2.1) is simplified owing to its symmetry and reduces to equality to zero of the saddle quantity $\sigma_c \equiv \lambda + \mu = 0$ in any of the saddles. In initial parameters this identity is equivalent to the equality

$$4a^2 \operatorname{tg}^2 \alpha + 4ab \operatorname{tg} \alpha + a^2 + b^2 - 1 = 0 \tag{2.2}$$

Neutral separatrix cycles in Eq.(0.1) can obviously form internal saddle separatrices (Fig.3,a), external saddle separatrices (Fig.3,b), and peripheral saddle separatrices that close into a loop (with the simultaneous appearance of four symmetric cycles (Fig.3,c).

In the investigation of curves of neutral separatrix cycles the following considerations expressed by V. I. Arnol'd are important. The parametric region in which neutral separatrix cycles are possible contain the line $a = 0$ on which for $\alpha = \pm\pi/2$ Eq.(0.1) is Hamiltonian. Hence the unknown cycles may be produced from neutral separatrix contours of the Hamiltonian equation (Fig.5). When this takes place line $a = 0$ intersects the curve of neutral separatric cycles. Such intersections can be determined by the standard technique of Hamiltonian equation perturbation.

Investigation of perturbations that maintain condition (2.2)has shown (see footnote on p.663) that line $a = 0$ along which Eq.(0.1) is Hamiltonian, is intersected only by curves of external neutral separatrix cycles. The intersection occurs for $b \approx \pm 4.1100817$. The same value of b was obtained in /4/ in which a complete analysis of Eq.(0.1) was carried out for parameter values close to Hamiltonian.

It is, thus, possible to state that the curve of external neutral separatrix cycles exists, and the problem is only to find it. Neutral separatrix cycles of the type shown in Fig.5,a,c do not apparently exist.

3. Numerical determination of nonlocal bifurcation curves. Each of the investigated curves is specified by two conditions. One of these is local and expressed in terms of eigenvalues of singular points. The second, nonlocal, corresponds to coincidence of separatrices. Explicit expression of the local condition makes it possible to reduce the number of parameters in the investigated equation and considerably reduce the procedure of bifurcation curve determination.

Curves *1* and *2*. Here (1.1) is the local condition. Using it we pass to the equation in which the peripheral singular points are saddle-nodes. For this we carry out in Eq.(0.1) the substitution

$$z = \frac{z_0}{1+i} w, \quad \rho = \frac{\sqrt{a^2 + b^2 - 1}}{2} \tag{3.1}$$

where z_0 is the singular point of the type of saddle-node $a^2 + b^2 > 1$.

The equation for w is of the form

$$w^{\cdot} = -\frac{e^{i\alpha}}{2} \left[w(|w|^2 - 2) + \frac{iw}{2\rho}(w^2 + \bar{w}^2) \right] \tag{3.2}$$

it contains two parameters: α and ρ, and the saddle-nodes are fixed at points $\pm 1 \pm i$.

Let S_1^- and S_2^+ be the ordinary and the singular separatrices of two adjacent saddle-nodes w_1 and w_2, and O_1 and O_2 be the intersection points of separatrices by the curve without contact L. The distance between points O_1 and O_2 defines the function $f(\alpha, \rho)$ of separatrix splitting. The problem is to determine parameters α and ρ that reduce function f to zero.

The calculation of function $f(\alpha, \rho)$ for arbitrary fixed parameters α and ρ reduces to the calculation of separatrices S_1^- and S_2^+. This is done by integrating Eq.(3.2). The initial points of integration are selected near saddle-nodes with allowance for separatrix asymptotics.

Let one of the points near the curve specified by the equation $f(\alpha, \rho) = 0$ be known. Then the process of determination of the curve is carried out by the method in /5/.

Note that one of the points on the curve that separates the generation of saddle-nodes outside and on it (curve *1*) was determined in /2/ using the condition that the joining separatrices are straight lines. Its coordinates are $\rho = 1, \alpha = -\pi/4$.

Curve *3*. The determination of neutral separatrix cycles is carried out similarly to that of curves *1* and *2*. Using the subsitution

$$z = \frac{z_0}{1+i} w, \quad \rho = \sin\alpha + \frac{b}{a}\cos\alpha$$

we pass from Eq.(1.1) to the equation

$$w^{\cdot} = -\frac{e^{i\alpha}}{2}\left[w(|w|^2-2) - \frac{1-e^{i\alpha}\rho}{2}\bar{w}(w^2+\bar{w}^2)\right] \qquad (3.3)$$

in which saddles are fixed at points $\pm 1 \pm i$ and the number of parameters is reduced by virtue of condition (2.2). The curve of separatrix cycles of Eq.(3.3) corresponds to the curve of neutral separatrix cycles of Eq.(0.1).

Note that the neutral separatrix cycles in Eq.(0.1) are exactly double. Proof of this statement consists of the verification of nondegeneracy of the principal term of the function of succession near the separatrix cycle(*). It was carried out numerically (points on the calculated curve *3* were spaced at a pitch of 0.01 with respect to parameter a). Stability of separatrix cycles on curve *3* for $a < 0$ and $-4.11 \ldots < b < -1$ was simulateously proved.

The numerical analysis of Eq.(0.1) was carried out on a computer, using the programs in /5-8/. The absolute error of calculation of curves *1*-*3* for $|b| \approx 1$ was 10^{-5} and for large $|b|$ the relative error was 10^{-3}. For $|b| \approx 4.11$ curve *3* was calculated with an accuracy to 10^{-4}.

5. Discussion. Of interest are the asymptotics of nonlocal bifurcation curves as $|a| \to 0$ and $|a| \to \infty$. The results of numerical analysis (detailed tables were given by the authors in the publication mentioned in the footnote on p. 663) make it possible to assume that the asymptotics are of quadratic form

for curve *1* $b \approx -1 + 0{,}47\,a^2$ as $b \to -1$, and $b \approx 0.35\,a^2$ as $b \to \infty$;

for curve *2* $b \approx -1 - 0.13\,a^2$ as $b \to -1$ and $b \approx -0{,}352\,a^2$ as $b \to -\infty$;

for curve *3* $b \approx -1 - 0.45\,a^2$ as $b \to -1$ and $b \approx -4.11 + 0.84\,a^2$ as $b \to -4.11$.

Note that, generally, one more bifurcation of codimension 2, the formation of a triple cylcle, could occur. The ends of the parametric curve corresponding to that bifurcation had to be sought at points that correspond to bifurcations of higher codimensions on line $a = 0$ and the line of neutral separatrix cycles. It was shown in /4/ that the "hypothetic" line of triple cycles does not intersect line $a = 0$. It was shown above that neutral separatrix cycles are exactly double. Hence there are no open lines of triple cycles in the parametric picture of Eq.(0.1).

The question of existence of closed lines of triple cycles, as well as that of existence of closed branches of remaining nonlocal bifurcation curves, remains open. We would, however, point out that numerical experiments make it possible to assume that such lines are absent.

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